Determination of Elastic and Piezoelectric Constants for Crystals in Class (3m)

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Determination of the elastic and piezoelectric constants for crystals in class (3m) is complicated by the large number of independent constants and the many possible ways in which they may be combined. An experimental and analytical procedure has been developed to determine all the constants using primarily thickness-mode measurements made on small, plate-shaped samples of various crystallographic orientations, and results using this procedure have been obtained for lithium tantalate and lithium niobate, two recently developed synthetic crystals. The resonant and antiresonant frequency constants for thickness modes have been calculated as functions of a plate's rotation angle. Information in this form makes possible the selection of plate orientations that might be useful as resonators and transducers.

INTRODUCTION

RECENTLY, synthetic crystals in the class (3m) have attracted much interest because of their unusual combination of ferroelectric, optical, elastic, and piezoelectric properties. The successful application of these materials in ultrasonic devices, whether as resonators for electromechanical filter applications or as transducers in devices such as ultrasonic delay lines, depends upon a knowledge of the complete set of elastic, piezoelectric, and dielectric constants. The primary objectives of the present paper are: (1) to present a combination of experimental and analytical techniques that makes possible the determination of all the elastic and piezoelectric constants, and (2) to show how the constants may be used to calculate the fundamental resonant and antiresonant frequencies of a thickness-mode plate vibrator as a function of the plate orientation and from this information to predict plate orientations of maximum usefulness. The methods of this paper are applied specifically to two materials of technological importance, lithium tantalate, LiTaO₃, and lithium niobate, LiNbO₃, that relatively small samples may be used and that the fabrication requirements are reduced to those of flatness, parallelism, and orientation of only the major faces of a sample. In principle it is possible to determine all the constants of materials in class (3m) by use of thickness modes, as has been done in the case of quartz, class (32), by Koga and Aruga. However, the ferroelectric materials considered in this paper exhibited a slight nonuniformity when the same measurement was made on different samples of the same material. Since the thickness mode frequencies are not very sensitive to certain constants, in particular ε₃₁ and ε₁₃, it was found necessary to make one additional measurement on a longitudinal mode resonator to aid in the determination of these constants.

The symbols used in this paper will be in accordance with the IRE Standards on Piezoelectric Crystals, 1949 and 1958. The most frequently used symbols are listed below:

- c elastic constant (stiffness)
- c' piezoelectrically stiffened elastic constant
- ε effective elastic constant (eigenvalue of Eq. 2)

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3 I. Koga and M. Aruga, “Theory of Plane Elastic Waves in a Piezoelectric Crystalline Medium and Determination of Elastic and Piezoelectric Constants of Quartz,” Phys. Rev. 109, 1467-1473 (1958). Their form of piezoelectrically stiffened constants is somewhat different from Eq. 3. In the case of quartz, neglecting the difference between ε₁₁ and ε₃₃, both formulas yield exactly the same results.

I. REVIEW OF THICKNESS MODES OF VIBRATION

A thickness mode of vibration in a plate can be interpreted as a standing wave formed by waves propagating in a direction normal to the major surfaces. The velocity $V_\alpha$ of a plane wave in a piezoelectric medium propagating in the direction $\mathbf{n}$ is given by the following equation:

$$V_\alpha = (c_\alpha / \rho)^{1/2}, \quad \alpha = 1, 2, 3,$$

where $c_\alpha$ is an effective elastic constant, which is one of three eigenvalues of the following secular determinant:

$$\lambda = 0,$$  

in which $G''$ is a piezoelectrically stiffened elastic constant and is given by

$$G'' = (\epsilon_{ij}^e / \rho)^{1/2}.$$  

If an effective elastic constant depends on piezoelectric constants, it is called piezoelectrically stiffened, while, if it does not, it is called unstiffened.

Unit eigenvectors $\beta_\alpha^{(\mathbf{n})}$ associated with $c_\alpha$ give the directions of the displacement vectors of the three waves.

The elastic wave motions existing in an electroded plate of a piezoelectric material are driven by an electric voltage applied to the electrodes. The determinantal equation for the resonant frequencies in the most general case, for a thickness-mode resonator excited by electrodes on the major surfaces, has been obtained by Tiersten and is given as

$$|\beta_\alpha^{(\mathbf{n})}|^2 c_{\mathbf{k}'}^e \gamma V_\alpha^{-1} \cos \gamma V_\alpha^{-1} - (\epsilon_{pi}^e \rho_{qi}^e \mu_{pi}^e \gamma_{qi}^e) (\epsilon_{ri}^e \rho_{ri}^e \gamma_{ri}^e)^{-1} \sin \gamma V_\alpha^{-1} = 0,$$

in which

$$\gamma = \pi f t.$$  

The insertion of a root back into one of the minor determinants of Eq. 4 yields the amplitude ratios of the waves contributing to the resonance. This determinantal equation is very complicated in general. For high overtones, however, Eq. 4 is closely approximated by

$$\cos \gamma V_\alpha^{-1} = 0.$$  

This simplification results because $\gamma$ is very large and appears only in the coefficient of the cosine term. Consequently, the use of high overtones is much preferred over the use of fundamental or low overtones for the purpose of determining the constants. It should also be noted that, while Eq. 6 is an approximation for the high overtone resonant frequencies, it is exact for the antiresonant frequencies of the fundamental as well as the overtone modes. On the other hand, the resonant frequencies for the first few resonances are shifted to lower values than those given by the above equation. However, because in a high overtone the difference between the resonant and the antiresonant frequency is small, the measurement of resonance instead of the more difficult measurement of antiresonance can safely be used.

Only stiffened waves can be excited when the electrodes are placed on the major surfaces. In this paper, this type of excitation is called perpendicular-field excitation. The unstiffened waves cannot be excited by the use of perpendicular-field excitation. On the other hand, some of them can be excited by a field parallel to the major surfaces. Such a field can be provided by placing the electrodes on the side (or minor) faces of a plate. This is called parallel-field excitation. For this type of excitation, the resonant frequencies are given exactly by Eq. 6. There is no frequency shift due to piezoelectric boundary conditions on the major surfaces even in the low order resonances. However, the effects of contour configuration on resonant frequencies are still noticeable in low order resonances. Hence the use of high order resonances is again preferable.
Equations 1 and 6 yield
\[ \bar{\varepsilon}_a = 4p (f_{m}^{\text{nth}})^2, \] (7)
where \( f_{m}^{\text{nth}} \) is the frequency of the \( m \)th overtone. Thus, in the most general case, three effective elastic constants for any given orientation may be obtained experimentally, since there are in general three independent wave motions with different phase velocities. The measurement of a series of high order resonances is recommended to obtain a positive identification of any one mode sequence by checking the harmonic relationship, and improved accuracy can be obtained by averaging of the data. The nature of these resonances and their overtones is illustrated in Fig. 1, which is actually an experimental plot of the resonances in a rotated \( \Gamma' \)-cut plate of lithium tantalate. The three distinct series of overtones are evident, along with resonances associated with the edge dimensions. The highest overtone modes are very close to the values of antiresonance. The resonances at lower overtones may differ from this value depending on the value of the electromechanical coupling factor, on whether parallel or perpendicular-field excitation is used, and on the ratio of the diameter to thickness dimensions of the plate. Although in principle any kind of plate orientation can be used, orientations that yield unstiffened modes are preferable, since the actual determination of the constants is simpler.

II. THICKNESS MODES FOR CRYSTAL PLATES IN CLASS (3m)

In the crystal class (3m), there are six independent elastic, four independent piezoelectric, and two independent dielectric constants, as shown in the elasto-piezo-dielectric matrix in Fig. 2. An examination of the secular determinant in Eq. 2 for this case reveals that several plate orientations yield at least one unstiffened mode.

A \( Z \)-cut, Fig. 3(a), yields two unstiffened pure shear modes with the same frequency constants, and one stiffened pure extensional mode. Any electric field direction parallel to the major surface can excite the unstiffened modes. An \( X \)-cut, Fig. 3(b), yields one unstiffened pure extensional mode and two stiffened shear modes. Any field direction parallel to the major surface can excite the unstiffened mode.

A rotated \( Y \)-cut, Fig. 3(d), which includes a \( Y \)-cut, Fig. 3(c), as a special case, yields one unstiffened pure shear mode and two stiffened modes, which are mixtures of shear and extensional motions. A field along the \( X \) axis excites the unstiffened mode and a field along \( Z' \) axis excites the remaining two stiffened modes. Hence an electrode configuration that gives only the field parallel to \( X \) axes is preferred because the identification of modes becomes simple. If the identification is not a problem, then the parallel-field electrodes may be rotated around the \( Y' \) or thickness axis, so that all three modes can be excited simultaneously with only one electrode configuration, as is the case in Fig. 1.

III. PROCEDURES FOR DETERMINING CONSTANTS

Dielectric constants can be obtained from capacitance measurement of plates with full electrodes. At frequencies well above any of the strong resonances, the constant \( e_{aa}^{\text{in}} \) is obtained from a \( Z \)-cut and \( e_{11}^{\text{in}} \) from either an \( X \)-cut or a \( Y \)-cut. At very low frequencies, well below any strong resonances, the constants \( e_{aa}^{\text{fr}} \) and \( e_{11}^{\text{fr}} \) are obtained. Although the constants \( e_{aa}^{\text{in}} \) and \( e_{11}^{\text{in}} \) are the ones needed to obtain the piezoelectric stress constants, the constants \( e_{aa}^{\text{fr}} \) and \( e_{11}^{\text{fr}} \) can be determined much more accurately from low-frequency capacitance measurements. To circumvent this problem, the experimental values for \( e_{aa}^{\text{br}} \) and \( e_{11}^{\text{br}} \) were used to obtain tentative values of the piezoelectric constants, and later all constants were readjusted slightly to fit the measured values of \( e_{aa}^{\text{fr}} \) and \( e_{11}^{\text{fr}} \).

Parallel-field excitation of unstiffened modes, as mentioned in the previous Section, immediately yields the following constants: \( e_{44}^{\text{br}} \) from a \( Z \)-cut, \( e_{11}^{\text{br}} \) from an \( X \)-cut, \( e_{66}^{\text{br}} \) from a \( Y \)-cut, and \( e_{11}^{\text{fr}} \) from a rotated \( Y \)-cut. Since the stiffened mode in a \( Z \)-cut is a pure extensional mode, the electromechanical coupling factor \( k = (e_{aa}^{\text{fr}}/c_{aa}^{\text{fr}}e_{aa}^{\text{fr}}) \) can be obtained from the ratios of measured fundamental and overtone resonant frequencies. This is the only fundamental thickness-mode resonant frequency needed for the measurement of the constants, so special selection of a large \( Z \)-cut plate free from unwanted resonances is desirable. The constants \( e_{aa}^{\text{fr}} \) and \( e_{11}^{\text{fr}} \) are obtained from \( k \) and the measured effective elastic constant \( \tilde{c}_1 \) by the following equations:

\[ e_{aa}^{\text{br}} = (1 - k^2)\hat{c}_3, \] (8)

\[ e_{11}^{\text{br}} = [e_{aa}^{\text{br}}\hat{d}_{3}x^2]^2. \] (9)

The sign of \( e_{aa}^{\text{br}} \) must be chosen so that the piezoelectric strain constant \( d_{33}^{\text{br}} \) is positive, as specified by the IRE standard on piezoelectric crystals. This usually, although not necessarily, implies that \( e_{aa}^{\text{br}} \) is positive.
(a) Z-cut
\[
\begin{vmatrix}
    c_{44} - c & 0 & 0 \\
    0 & c_{44} - c & 0 \\
    0 & 0 & c_{55} + \beta_{11} e_{15}^2 - c
\end{vmatrix} = 0
\]

(b) X-cut
\[
\begin{vmatrix}
    c_{11} - c & 0 & 0 \\
    0 & c_{44} + \beta_{11} e_{15}^2 - c & c_{14} - \beta_{11} e_{15} e_{22} \\
    0 & c_{14} - \beta_{11} e_{15} e_{22} & c_{44} + \beta_{11} e_{15}^2 - c
\end{vmatrix} = 0
\]

(c) Y-cut
\[
\begin{vmatrix}
    c_{44} - c & 0 & 0 \\
    0 & c_{44} + \beta_{11} e_{15}^2 - c & -c_{14} - \beta_{11} e_{15} e_{22} \\
    0 & -c_{14} + \beta_{11} e_{15} e_{22} & c_{44} + \beta_{11} e_{15}^2 - c
\end{vmatrix} = 0
\]

(d) Rotated Y-cut (around X-axis)
\[
\begin{vmatrix}
    L & 0 & 0 \\
    0 & M' - c & F' \\
    0 & F' & N' - c
\end{vmatrix} = 0
\]

where
\[
L = m^2 c_{44} + n^2 c_{44} + 2 m n c_{44}, \\
M' = m^2 c_{11} + n^2 c_{11} + 2 m n c_{11} + \beta (m e_{15} + m e_{15} + n e_{15})^2, \\
N' = m^2 c_{55} + n^2 c_{55} + \beta (m e_{15} + n e_{15})^2, \\
F' = m^2 (c_{11} + c_{44}) - m^2 e_{15} + \beta (m e_{15} + m e_{15} + n e_{15}) (m e_{15} + n e_{15}).
\]

Thus, since all constants appearing in Eqs. 14–17 other than \(e_{15}\) and \(e_{22}\) are known, we can determine the magnitude of \(e_{15}\) from Eq. 14, the magnitude of \(e_{22}\) from both Eqs. 15 and 16, and the relative sign between \(e_{15}\) and \(e_{22}\) from Eq. 17. The absolute signs of \(e_{15}\) and \(e_{22}\) are selected so that the piezoelectric strain constant \(d_{22}\) is positive. That is,
\[
d_{22} = e_{22} (s_{11}^E - s_{15}^E) - e_{15} s_{14}^E > 0,
\]
according to the IRE convention. Notice that Eqs. 14–17 provide four equations to solve for only two unknowns, so that any inconsistencies in the measurements are immediately apparent.

Perpendicular-field excitation of a rotated Y-cut yields two effective elastic constants. From Fig. 3 their sum is expressed by the following equation:
\[
(\tilde{c}_{1} + \tilde{c}_{3})_{Y'} = M' + N',
\]
in which all constants except \(e_{31}\) are known. Since Eq. 19 is a quadratic equation for \(e_{31}\), selection of the proper value for \(e_{31}\) can in principle be made by comparing the values obtained from two different rotated Y-cuts.

Unfortunately the values of \(e_{31}\) obtained from Eq. 19 are extremely sensitive to the measured values of \(c_{31}\) and \(c_{33}\) and small errors in these effective elastic constants can lead to fairly large errors in \(e_{31}\). Hence it is desirable to measure some quantity that depends more strongly on \(e_{31}\), and such a quantity is the coupling factor \(k_{31}\) of a rectangular bar with its length along the X axis and with electrodes applied to the Z faces. When the length of the bar is much larger than the transverse dimensions, the coupling factor \(k_{31}\) is approximately given by
\[
k_{31} = (d_{31}^2 / e_{32}^2 e_{31}^3 b^3)^{1/3}.
\]
The sign of \(d_{31}\) can be determined by a static test, and since \(s_{15}^E\) can be found from the fundamental resonant frequency of the bar, \(d_{31}\) can be calculated from Eq. 20. Then the equation
\[
e_{31} = d_{31} (c_{44}^E + c_{66}^E - 2 (c_{14}^E)^2 / c_{33}^E) + c_{14}^E e_{22}^E / c_{33}^E
\]

By a careful rearrangement of Eqs. 10–13, we find that
\[
c_{44}^E + \beta_{11} s_{15}^2 e_{15}^2 = (\Pi Y - \Pi X) / (\Pi Y - \Sigma X),
\]
\[
c_{66}^E + \beta_{11} s_{22}^2 e_{22}^2 = \Sigma X - (\Pi Y - \Pi X) / (\Pi Y - \Sigma X),
\]
\[
c_{11}^E + \beta_{11} s_{15}^2 e_{22}^2 = \Sigma Y - (\Pi Y - \Pi X) / (\Pi Y - \Sigma X),
\]

(\(e_{32}\) is, of course, different from the one obtained from a Z-cut in the previous section. When it is necessary, the orientation of a plate will be identified in the following manner: \((e_{32})_Y\) and \((e_{32})_Z\), etc.)
provides a relation between the two remaining unknown constants $\varepsilon_3$ and $\varepsilon_2$.

The magnitude of $\varepsilon_2$ can be found from the value of $s_{11}$ found from the bar measurement, with the sign of $\varepsilon_2$ selected to give agreement with the effective elastic constants measured on a rotated Y-cut plate. Alternatively, $\varepsilon_3$ can be selected to give the best fit to measurements made on different rotated Y-cut plates and in this way obtain a check on the consistency of the data.

### IV. EXPERIMENTAL TECHNIQUES

The plates used in the experiments were irregular in shape and averaged in area about 0.25 cm$^2$. The small size is a consequence of using small, experimentally grown boules of crystalline material. The ability to use such plates is of course a distinct advantage in the evaluation of new materials. Every effort was made to produce flat, parallel plates so that the thickness dimension would have significance to at least three figures. In practice, the thickness ranged from 0.200 to 1.000 mm with thickness variations in a given plate less than 0.0005 mm ($\frac{1}{4}$ μ). The thicker plates could be operated on high overtones, while the thinner plates gave better freedom from unwanted modes at the fundamental and low overtone modes.

The electrodes used were gold, deposited directly on the crystal plates by evaporation in a vacuum of about 10$^{-4}$ torr. Figure 4 shows the electrode configuration for perpendicular- and parallel-field excitation. The perpendicular-field plates were clamped directly in a Wayne-Kerr 1-100-MHz admittance bridge to minimize lead inductance, while the parallel-field plates were held in the bridge by miniature spring clips. In some instances, where a desired resonance was extremely weak, additional sensitivity was obtained by the use of a hybrid transformer bridge such as is used to evaluate unwanted resonances in quartz filter plates. The admittance bridge is, of course, more desirable because of its better definition of the series resonance.

The use of a sweep oscillator was found to be extremely useful in sorting out the many resonances in any one plate, as well as in selecting a frequency free from resonances for the measurement of capacitance. An extreme example is shown in Fig. 1 where even and odd overtones of the thickness extensional mode and the odd overtones of both thickness shear modes can be identified. It can further be seen that some estimate of coupling and $Q$ can be made as well as an identification of the several series of resonances.

Figure 5 shows the schematic diagram of the system for measuring the series resonant frequencies of a plate. The sweep oscillator is continuously variable as to sweep width, sweep rate, and center frequency. Crystal controlled harmonic markers are provided every 1, 2, 5, or 10 MHz. It is usually sufficient in the measurement of any one resonance to set the sweep for a narrow range of frequency, set sweep rate to manual, find the peak or null depending on which bridge is in use, and read the frequency on a frequency counter. If greater precision is desirable, a frequency synthesizer may be substituted for the sweep oscillator, and a tuned voltmeter for the detector. The system covers from 50 KHz to 100 MHz, and since the fundamental resonance of a plate may be near 3 MHz, overtones as high as 30 can be measured.

The accuracy of determining the frequency constant from overtones is about ±0.1% limited principally by the thickness measurement. Since values from a number of plates of lithium tantalate of the same orientation show variations as high as 1%, particularly when different boules or crystals are involved, it is believed that imperfections in domain structure are significant in this measurement.

![Fig. 5. System for measuring resonant frequencies.](image)

<table>
<thead>
<tr>
<th><strong>Table I. Constants of lithium tantalate.</strong></th>
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<td><strong>Dielectric constants</strong></td>
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30 R. L. Barns, “X-ray Powder Data, Density and Precision Lattice Parameters of Lithium Tantalate, LiTaO$_3$” (to be published).
Comparison between measured and calculated frequency constants of lithium tantalate (unit: hertz-meter).

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V. DETERMINATION OF CONSTANTS FOR LITHIUM TANTALATE

Single domain crystals of lithium tantalate, LiTaO₄ are grown by the Czochralski technique. X-, Y- and Z-cuts as well as several rotated Y-cuts are fabricated and the orientations are checked by x-ray methods.

The procedures from the first three paragraphs in Sec. III are straightforward. The coupling factor k₁ of the Z-cut plate is 19%. In the procedure of the fourth paragraph the following results were obtained for ϵ₁₁ and ϵ₂₂ from Eqs. 14-17:

$$\epsilon_{11}^2 = 6.8 \text{ C/m}^2$$
$$\epsilon_{22}^2 = 3.1$$
$$\epsilon_{33}^2 = 2.1$$
$$\epsilon_{11}\epsilon_{22} = -4.1$$

The above results show a fair amount of inconsistency and the values ϵ₁₁ = 2.6 and ϵ₂₂ = 1.6 were selected as a compromise fit. In the procedure of the sixth paragraph of Sec. III the coupling factor k₃₃ was found to be 8%, and the value ϵ₁₁ = -0.2×10⁻¹¹ C/N is obtained from this. When the value of ϵ₁₁ found by the procedure of the seventh paragraph is used, the value of ϵ₁₃ calculated from Eq. 21 is ϵ₁₃ = 0.0 to two significant figures. This does not at all imply that ϵ₁₃ is exactly equal to zero.

The results are summarized in Table I. With these constants, it is possible to calculate the frequency constants of the fundamental resonances from Eq. 5 for plates of any orientation. Table II shows the comparison between measured and calculated frequency constants for the fundamental antiresonance (high order resonance) and resonance of various cuts. Good experimental accuracy for the fundamental resonance should not be expected because of the small size of the plates. In some instances the fundamental resonance could not be measured due to interference of other modes. The agreement between measured and calculated frequency constants seen in Table II is reasonably good, with discrepancies less than 1% in most cases. Figures 6-8 show the variation of frequency constants of the fundamental resonances and antiresonances of a lithium-tantalate plate, when it is excited by a perpendicular field, as functions of rotation angle of a plate around the X, Y, and Z axes, respectively. The separation between the resonance and antiresonance of a stiffened mode is a measure of the strength of a mode. The figures show no such separation for an unstiffened mode, because it is not excited by a perpendicular field.
DETERMINATION OF CONSTANTS FOR CRYSTALS

We can define an effective coupling factor for a mode in terms of the separation between the fundamental resonance and antiresonance as follows:

$$k_{eff} = \left[ \frac{\pi f_r}{\tan \frac{\pi f_r}{2 f_x}} \right],$$  \hspace{1cm} (22)

where $f_r$ and $f_x$ are the resonant and antiresonant frequencies, respectively. This definition is equivalent to the usual definition of coupling factor when there is only one stiffened mode. The effective coupling factors of the quasishear and quasiextensional modes of a rotated $Y$-cut plate are plotted in Fig. 9 as functions of the angle of rotation. Also plotted in Fig. 9 is the angle $\phi$ between the extensional wave displacement vector and the plate normal. Since the displacement of the unstiffened shear wave is always along the $X$ axis, $\phi$ is also the angle the stiffened shear wave displacement makes with the plane of the plate.

For transducer applications it is advantageous to have a high effective coupling factor, but in addition it is often required that only one wave, extensional or shear, be excited. Referring to Fig. 9, we can see that for the 165° rotated $Y$-cut plate the effective coupling factor of the quasiextensional mode vanishes whereas the effective coupling factor of the quasishear mode has a high value of 41%. Also the angle $\phi$ is nearly zero so that the mode of vibration is nearly a pure mode. Hence this cut would make an excellent shear wave transducer. Similarly, the 47° cut has a quasiextensional mode coupling of 29% and no coupling to the quasishear mode. The angle $\phi$ for this cut is 1.4° which, although not as small as for the 165° cut, is small enough so that this cut could be used as an extensional wave transducer for most applications. The 111° cut also has no coupling to the quasishear mode, but $\phi = -2.6°$ for this cut, and this is too large to permit use as a transducer because an excessive amount of shear wave would be excited. Of course the $Z$-cut, with a coupling factor of 19%,

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**Fig. 8.** Frequency constants of the fundamental resonances and antiresonances of a LiTaO$_3$ plate rotated around the $Z$ axis.

**Fig. 9.** Effective coupling factors and angle $\phi$ between quasiextensional wave displacement and plate normal for rotated $Y$-cuts of LiTaO$_3$.

**Fig. 10.** Frequency constants of the fundamental resonances and antiresonances of a LiNbO$_3$ plate rotated around the $X$ axis (rotated $Y$-cut).

**Fig. 11.** Frequency constants of the fundamental resonances and antiresonances of a LiNbO$_3$ plate rotated around the $Y$ axis.
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RESONANT FREQUENCY (-) AND ANTIRESONANT FREQUENCY (+) ROTATED AROUND Z AXIS
CENTER IS Y-CUT

![Diagram of resonant frequency and antiresonant frequency for Y-cut and X-cut LiNbO₃ plates rotated around the Z axis.]

FIG. 12. Frequency constants of the fundamental resonances and antiresonances of a LiNbO₃ plate rotated around the Z axis.

could also be used as an extensional wave transducer. All of the cuts mentioned above could be used for resonator applications, since in that case the purity of the mode of vibration is immaterial. Notice that in all cases the effective coupling factor goes to zero linearly with the angle of rotation rather than quadratically. This implies that the mode whose effective coupling goes to zero is excited with opposite phase in rotated Y-cuts with angles on either side of the angle at which the effective coupling is zero.

Another interesting point, which can be observed in Figs. 7 or 8, is that one of the stiffened shear modes in an X-cut plate is very weak. The other shear mode is quite strong with an effective coupling factor of 44%. An X-cut plate used as a transducer would excite one shear wave in the delay medium very strongly and the other shear wave would be weakly excited. Since the two shear waves in an isotropic delay medium are degenerate, this is not objectionable. Thus the X-cut plate, because of its high effective coupling factor,
Determination of Constants for Crystals

Table IV. Comparison between measured and calculated frequency constants of lithium niobate (unit: hertz-meter).

<table>
<thead>
<tr>
<th></th>
<th>Unstiffened</th>
<th>Antiresonance</th>
<th>Resonance</th>
<th>Stiffened</th>
<th>Antiresonance</th>
<th>Resonance</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>mes. cal.</td>
<td>mes. cal.</td>
<td>mes. cal.</td>
<td>mes. cal.</td>
<td>mes. cal.</td>
<td>mes. cal.</td>
</tr>
<tr>
<td>X</td>
<td>3290 3289</td>
<td>2405 2397</td>
<td>2041 2032</td>
<td>2032 2034</td>
<td>1838 1803</td>
<td>1803</td>
</tr>
<tr>
<td>Y</td>
<td>1993 1993</td>
<td>3428 3430</td>
<td>3297 3306</td>
<td>2239 2231</td>
<td>1868 1845</td>
<td>1845</td>
</tr>
<tr>
<td>Z</td>
<td>1788 1788</td>
<td>3660 3659</td>
<td>3613 3620</td>
<td>2032 2034</td>
<td>1788</td>
<td>1788</td>
</tr>
<tr>
<td>Rotated</td>
<td>45°  2012 2015</td>
<td>3692 3661</td>
<td>3305 3342</td>
<td>2023 1994</td>
<td>1803 1776</td>
<td>1776</td>
</tr>
<tr>
<td></td>
<td>60°  1942 1950</td>
<td>3664 3629</td>
<td>3415 3467</td>
<td>1933 1935</td>
<td>1900 1871</td>
<td>1871</td>
</tr>
<tr>
<td></td>
<td>135°  1750 1763</td>
<td>3531 3542</td>
<td>3438 3436</td>
<td>2067 2041</td>
<td>2052 1999</td>
<td>1999</td>
</tr>
<tr>
<td></td>
<td>160°... 1891...</td>
<td>... 3366...</td>
<td>... 3363 2276 2244</td>
<td>1896 1869</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

quasishear wave, but the angle $\zeta$ in both cases is too large for these cuts to be useful for transducers, and only the Z-cut is suitable for an extensional wave transducer.

As in the case of lithium tantalate, one of the stiffened shear modes in an X-cut plate is very weak. The other shear mode in this case has an effective coupling factor of 68%, thus the X-cut would make an excellent shear wave transducer on an isotropic delay medium. It can be seen in Figs. 10 and 11 that the frequency constants of the weak shear mode lie between the resonance and antiresonance of the strong shear mode. This causes the interesting phenomenon of the weak mode having an antiresonance lower in frequency than its resonant frequency, and the interesting phenomenon of the weak mode having an antiresonance lower in frequency than its resonant frequency, since it is obvious that a crystal cannot have two resonances without an intermediate antiresonance.

VII. CONCLUSION

A method for determining all the elastic and the piezoelectric constants of a crystal in the class $(\bar{3}m)$ has been discussed. By making use primarily of resonant frequencies of high overtones in thin plates, this method allows the use of rather small crystals.

The constants of lithium niobate and lithium tanta-

late have been determined. The variation of frequency constants of the fundamental resonance as well as antiresonance have been calculated as functions of rotation angle around the X, Y, and Z axes, so that useful cuts may be selected. Several cuts of both materials, and in particular lithium niobate because of its very high effective coupling factors, appear to be useful for transducer applications.

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